Skin Effect Description in Electromagnetism with a Multiscaled Asymptotic Expansion

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Skin-Effect Description in Electromagnetism with a Scaled Asymptotic Expansion

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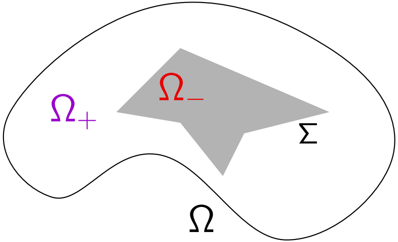
References

- GABRIEL CALOZ, MONIQUE DAUGE, VICTOR PÉRON (2009)
  Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism
- MONIQUE DAUGE, ERWAN FAOU, VICTOR PÉRON (2009)
  Asymptotic Behavior at High Conductivity of Skin Depth in Electromagnetism

Transmission Problems with High Contrast Media

ELECTROMAGNETISM

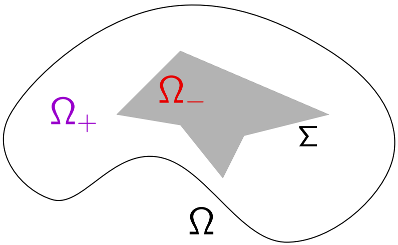


- Ω_- Highly conducting material $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or dielectric body: Conductivity $\sigma_+ = 0$

Aim: Understand the electromagnetic phenomenon as $\sigma \to \infty$

Transmission Problems with High Contrast Media

HEAT CONDUCTION



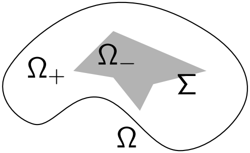
- $\Omega_- \subset\subset \Omega$: heat conductivity a_-
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+: heat conductivity a_+

Aim: Describe the transmission phenomenon as $\left|a_- / a_+\right| \gg 1$

Outline

Outline

Transmission Problems in High Contrast Media



- Heat transfer equation $(\mathbf{P}_{\underline{a}})$: $\quad \operatorname{div} \underline{a}\, \mathbf{grad}\, \varphi = f$

$$\underline{a} = (a_+, a_-) \quad \text{s.t.} \quad |a_-|/|a_+| \to \infty$$

- Maxwell equations $(\mathbf{P}_{\underline{\sigma}})$:
$\operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad$ and $\quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j}$

$$\underline{\sigma} = (\sigma_+, \sigma_-) \quad \text{s.t.} \quad \sigma_+ = 0 \quad \text{and} \quad \sigma_- \equiv \sigma \to \infty$$

Issues

1. *Uniform piecewise estimates and regularity* in Sobolev norms for solutions φ of $(\mathbf{P}_{\underline{a}})$
2. *Uniform* L^2 *estimates* for solutions $(\mathbf{E}, \mathbf{H})$ of $(\mathbf{P}_{\underline{\sigma}})$

Hypothesis

Σ *is a bounded Lipschitz surface in* $\mathbb{R}^3$

Scalar case:

- H.P. Huy, E. Sanchez-Palencia (1974)
  Limit problem and Strong convergence results as $a_- / a_+ \to \infty$
- S. Hassani, S. Nicaise, A. Maghnouji (2009)

Maxwell case:

- H. Haddar, P. Joly, H.N. Nguyen (2008)
  Uniform estimates under a stronger regularity assumption on Σ

Scalar Problem

Variationnal Problem

$$V_{\mathrm{D}} = \mathrm{H}^1_0(\Omega) \quad \text{or} \quad V_{\mathrm{N}} = \{\varphi \in \mathrm{H}^1(\Omega) \mid \int_\Omega \varphi \,\mathrm{d}\mathbf{x} = 0\}$$

$(\mathbf{VP}_{\underline{a}})$: Find $\varphi \in V = V_{\mathrm{D}}$ or $V = V_{\mathrm{N}}$ such that

$$\forall \psi \in V, \quad \int_{\Omega_+} a_+ \nabla\varphi^+ \cdot \nabla\overline{\psi}^+ \,\mathrm{d}\mathbf{x} + \int_{\Omega_-} a_- \nabla\varphi^- \cdot \nabla\overline{\psi}^- \,\mathrm{d}\mathbf{x} = - \int_\Omega f\, \overline{\psi} \,\mathrm{d}\mathbf{x} + (a_+ - a_-) \int_\Sigma g\, \overline{\psi} \,\mathrm{d}s$$

Hypothesis $(\mathrm{H}_{\underline{a}})$

1. $f \in \mathrm{L}^2(\Omega)$ *and* $g \in \mathrm{L}^2(\Sigma)$
2. $\int_\Omega f \,\mathrm{d}\mathbf{x} = \int_\Sigma g \,\mathrm{d}s = 0$ *if* $V = V_{\mathrm{N}}$ *and* $\int_\Sigma g \,\mathrm{d}s = 0$ *if* $V = V_{\mathrm{D}}$

Uniform Estimates

Theorem

Let us assume that $a_+ \neq 0$. *There exist* $\rho_0 > 0$ *independent of* a_+ *such that for all* $a_- \in \{z \in \mathbb{C} \mid |z| \geqslant \rho_0 |a_+|\}$, *and for all data* (f, g) *satisfying* $(\mathbf{H}_{\underline{a}})$, $(\mathbf{VP}_{\underline{a}})$ *has a unique solution* $\varphi \in V$, *which is piecewise* $\mathrm{H}^{3/2}$ *and*

$$\|\varphi^+\|_{\frac{3}{2},\Omega_+} + \|\varphi^-\|_{\frac{3}{2},\Omega_-} \leqslant C_{\rho_0}\left(|a_+|^{-1}\|f\|_{0,\Omega} + \|g\|_{0,\Sigma}\right)$$

with $C_{\rho_0} > 0$, *independent of* a_+, a_-, f, *and* g.

GABRIEL CALOZ, MONIQUE DAUGE, V. PÉRON (2009)

Key of the proof: asymptotic expansion for φ with respect to the powers of $\rho^{-1} := a_+(a_-)^{-1}$ and *convergence* of these series in the piecewise $\mathrm{H}^{3/2}$-norm

Remarks

1. Similar estimate when the roles of a_+ and a_- are exchanged. More precise estimate where a_+ and a_- play symmetric roles
2. The assumption that Σ is Lipschitz is necessary
3. For *polyhedral* Lipschitz interface Σ, uniform piecewise H^s estimates:

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leqslant C_{\rho_0}\left(|a_+|^{-1}\|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma}\right)$$

   for $s \leqslant s_\Sigma$ with some $\frac{3}{2} < s_\Sigma \leqslant 2$
4. For *smooth* interface Σ, uniform piecewise H^s estimates for any $s \geqslant 2$

Maxwell Problem

Framework and Boundary Conditions

$$(\mathbf{P}_{\underline{\sigma}}) \qquad \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j} \qquad \underline{\sigma} = (0, \sigma)$$

1. $\mathbf{E}\times\mathbf{n} = 0$ and $\mathbf{H}\cdot\mathbf{n} = 0$ on $\partial\Omega$
2. $\mathbf{E}\cdot\mathbf{n} = 0$ and $\mathbf{H}\times\mathbf{n} = 0$ on $\partial\Omega$

1. B. C. 1: $\mathbf{j} \in \mathrm{H}(\operatorname{div},\Omega) = \{\mathbf{u}\in\mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{u}\in \mathrm{L}^2(\Omega)\}$
2. B. C. 2:
   $\mathbf{j} \in \mathrm{H}_0(\operatorname{div},\Omega) = \{\mathbf{u}\in\mathrm{H}(\operatorname{div},\Omega) \mid \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$

Hypothesis (*SH*)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \textit{and} \quad \operatorname{curl}\mathbf{H} + i\omega\varepsilon_0\mathbf{E} = 0 & \textit{in} \quad \Omega_+ \\ \mathbf{E}\times\mathbf{n} = 0 \quad \textit{and} \quad \mathbf{H}\cdot\mathbf{n} = 0 & \textit{on} \quad \Sigma \\ B.C.1 \quad \textit{or} \quad B.C.2 & \textit{on} \quad \partial\Omega \end{cases}$$

Maxwell Problem

Uniform $\mathbf{L}^2(\Omega)$ Estimate

Theorem

Under Hypothesis (SH), there are σ_0 and $C > 0$, such that for all $\sigma \geqslant \sigma_0$, the Maxwell problem $(\mathbf{P}_{\underline{\sigma}})$ with B.C.2 and $\mathbf{j} \in \mathrm{H}_0(\mathrm{div}, \Omega)$ has a unique solution $(\mathbf{E}, \mathbf{H})$ in $\mathbf{L}^2(\Omega)^2$, which satisfies:

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma}\, \|\mathbf{E}\|_{0,\Omega_-} \leqslant C\|\mathbf{j}\|_{\mathrm{H}(\mathrm{div},\Omega)}$$

Similar result for B.C.1 and $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$

Gabriel Caloz, Monique Dauge, V. Péron (2009)

Application: Convergence of asymptotic expansion at high conductivity

Maxwell Problem

Proof of Maxwell Uniform Estimate

Lemma

Under Hypothesis (SH), *there are* σ_0 *and* $C_0 > 0$ *such that if* $\sigma \geqslant \sigma_0$ *any solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\Omega)^2$ *of problem* $(\mathbf{P}_{\underline{\sigma}})$ *with B.C.2 and data* $\mathbf{j} \in \mathrm{H}_0(\mathrm{div}, \Omega)$ *satisfies the estimate*

$$\|\mathbf{E}\|_{0,\Omega} \leqslant C_0 \|\mathbf{j}\|_{\mathrm{H}(\mathrm{div},\Omega)}$$

Similar statement for B.C.1 and $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$.

Keys of the proof:

- Electrical Field Decomposition into a regular field and a gradient field

  C. Amrouche, C. Bernardi, M. Dauge, V. Girault (1998)

- Scalar Estimates used for the gradient field $\nabla\varphi$

Outline

References and Notations

Asymptotic expansion as $\sigma \longrightarrow \infty$ of solutions of $(\mathbf{P}_{\underline{\sigma}})$ *when* Σ *is smooth*:

- E. Stephan, R.C. McCamy (1983-84-85)
  Plane Interface and Eddy Current Approximation
- H. Haddar, P. Joly, H.N. Nguyen (2008)
  Smooth Interface and Impedance boundary conditions
- V. Péron (2009)
  Smooth Interface and Perfectly Conducting Electric or Magnetic Wall B.C.

Hypothesis

1. Σ *is an oriented smooth compact surface*
2. *Perfectly Conducting Magnetic Wall b.c.*
3. $\mathbf{j} \in \mathrm{H}_0(\mathrm{div}, \Omega)$ *and* $\mathbf{j} = 0$ *in* Ω_-
4. *Hypothesis (SH) on* ω

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega\varepsilon_0}{\sigma}} \to 0 \quad \text{as} \quad \sigma \to \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leqslant \delta_0$, there exists a unique solution $\big(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)}\big)$ to $\big(\mathbf{P}_{\underline{\sigma}}\big)$

Then it is possible to construct series expansions in powers of δ:

$$\mathbf{E}^{+}_{(\delta)}(\mathbf{x}) \;\approx\; \sum_{j\geqslant 0} \delta^j \mathbf{E}^{+}_j(\mathbf{x}), \qquad \mathbf{E}^{-}_{(\delta)}(\mathbf{x}) \;\approx\; \chi(y_3) \sum_{j\geqslant 0} \delta^j \mathbf{W}_j\big(y_\beta, \frac{y_3}{\delta}\big)$$

$$\mathbf{E}^{+}_j \in \mathrm{H}(\mathrm{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathrm{H}(\mathrm{curl}, \Sigma \times \mathbb{R}_+) \quad \textit{profiles}$$

(y_β, y_3): "*normal coordinates*" to Σ in a tubular neighborhood $\mathcal{U}_-$ of Σ in Ω_-

Similar result for the magnetic field.

Validation of the asymptotic expansion

Remainders

Aim: proving estimates for remainders $\mathbf{R}_{m;\,\delta}$

$$\mathbf{R}_{m;\,\delta} := \mathbf{E}_{(\delta)} - \sum_{j=0}^{m} \delta^j \mathbf{E}_j \quad \text{in} \quad \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\,\delta}$:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{R}^+_{m;\,\delta} - \kappa^2\alpha_+\mathbf{R}^+_{m;\,\delta} & = & 0 & \text{in} & \Omega_+ \\ \operatorname{curl}\operatorname{curl}\mathbf{R}^-_{m;\,\delta} - \kappa^2\alpha_-\mathbf{R}^-_{m;\,\delta} & = & \mathbf{j}^-_{m;\,\delta} & \text{in} & \Omega_- \\ \left[\mathbf{R}_{m;\,\delta}\times\mathbf{n}\right]_\Sigma & = & 0 & \text{on} & \Sigma \\ \left[\operatorname{curl}\mathbf{R}_{m;\,\delta}\times\mathbf{n}\right]_\Sigma & = & \mathbf{g}_{m;\,\delta} & \text{on} & \Sigma \\ \operatorname{curl}\mathbf{R}^+_{m;\,\delta}\times\mathbf{n} & = & 0 & \text{on} & \partial\Omega \end{cases}$$

with $\alpha_+ = 1$ and $\alpha_- = 1 + i/\delta^2$

Validation of the asymptotic expansion

Estimates for Remainders

$$\|\mathbf{j}^{-}_{m;\,\delta}\|_{2,\Omega_-} + \|\mathbf{g}_{m;\,\delta}\|_{\frac{1}{2},\Sigma} + \|\operatorname{curl}_\Sigma \mathbf{g}_{m;\,\delta}\|_{\frac{3}{2},\Sigma} \leqslant C_m \delta^{m-1}$$

with $C_m > 0$ independent of δ.

Theorem

Under Hypothesis in the framework above, for all $m \in \mathbb{N}$ and $\delta \in (0, \delta_0]$, the remainders $\mathbf{R}_{m;\,\delta}$ satisfy the optimal estimates

$$\|\mathbf{R}^{+}_{m;\,\delta}\|_{0,\Omega_+} + \|\operatorname{curl} \mathbf{R}^{+}_{m;\,\delta}\|_{0,\Omega_+} + \delta^{-\frac{1}{2}}\|\mathbf{R}^{-}_{m;\,\delta}\|_{0,\Omega_-} + \delta^{\frac{1}{2}}\|\operatorname{curl} \mathbf{R}^{-}_{m;\,\delta}\|_{0,\Omega_-} \leqslant C'_m \delta^{m+1}$$

Electrical Profiles

$\mathbf{W}_j =: (\mathcal{W}_{\alpha,j}\ ;\ w_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

$$\mathbf{W}_0 = 0$$

$$\mathcal{W}_{\alpha,1} = -j_{\alpha,0}(y_\beta)\,\mathrm{e}^{-\lambda Y_3} \quad \text{and} \quad w_1 = 0$$

$$\mathcal{W}_{\alpha,2} = \Big[\lambda^{-1}\big\{b_\alpha^\sigma\, j_{\sigma,0} - \mathcal{H}\, j_{\alpha,0}\big\} - j_{\alpha,1} + Y_3\big\{b_\alpha^\sigma\, j_{\sigma,0} - \mathcal{H}\, j_{\alpha,0}\big\}\Big](y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

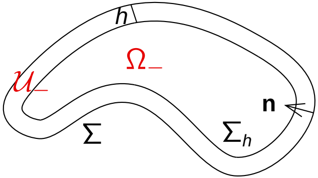
$$w_2 = -\lambda^{-1} D_\alpha\, j_0^\alpha(y_\beta)\,\mathrm{e}^{-\lambda Y_3}$$

with

$$\mathcal{H} \quad \text{mean curvature of } \Sigma$$

$$j_{\alpha,k}(y_\beta) := \lambda^{-1}(\mathrm{curl}\,\mathbf{E}_k^+ \times \mathbf{n})_\alpha(y_\beta, 0) \quad \text{and} \quad \lambda = \kappa\,\mathrm{e}^{-i\pi/4}$$

Curl Operator and Levi-Civita Tensor



Levi-Civita Tensor ϵ:

$$\epsilon^{ijk} = \epsilon_0(i,j,k) \Big/ \sqrt{\det\left(a_{\alpha\beta}(h)\right)}$$

$a_{\alpha\beta}(h)$: metric on Σ_h and $\epsilon_0(i,j,k) = $ sign of (i,j,k)

Curl Operator in Normal Parameterization:

$$\begin{cases} (\nabla \times \mathbf{E})^\alpha = \epsilon^{3\beta\alpha}(\partial_3^h E_\beta - \partial_\beta E_3) & \text{on} \quad \Sigma_h \\ (\nabla \times \mathbf{E})^3 = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta & \text{on} \quad \Sigma_h \end{cases}$$

D_α^h : Covariant Derivative on Σ_h

Magnetical Profiles

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha \; ; \; v_j)$ in (y_β, Y_3) coordinates

$$\mathbf{V}_0(y_\beta, Y_3) = H_0(y_\beta)\, \mathrm{e}^{-\lambda Y_3} \,,$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \Big[H_1^\alpha(y_\beta) + Y_3 \Big(\mathcal{H}\, H_0^\alpha + b_\sigma^\alpha \, H_0^\sigma \Big)(y_\beta) \Big] \, \mathrm{e}^{-\lambda Y_3} \,,$$

$$v_1(y_\beta, Y_3) = \lambda^{-1} \mathbf{D}_\alpha \, H_0^\alpha(y_\beta)\, \mathrm{e}^{-\lambda Y_3}$$

with $H_j^\alpha(y_\beta) := (\mathbf{n} \times \mathbf{H}_j^+) \times \mathbf{n}^\alpha(y_\beta, 0)$

Comparison with

H. HADDAR, P. JOLY, H.N. NGUYEN (2008)

Skin Effect and Skin Depth

1D model of *Skin Effect* : $\{z > 0\}$ = half-space conductor body

$$\|\mathbf{E}_\sigma(z)\| = E_0\, \mathrm{e}^{-\frac{z}{\ell(\sigma)}}\,, \quad \ell(\sigma) := \sqrt{\frac{2}{\omega\mu_0\sigma}}$$ *Skin Depth*

$\ell(\sigma) = \delta / \operatorname{Re}\lambda\,.$

3D model of *Skin Effect*, using *profiles* $\mathbf{V}_0$, $\mathbf{V}_1$,

$$\mathbf{V}(y_\alpha, y_3) := \mathbf{V}_0(y_\alpha, \frac{y_3}{\delta}) + \delta\mathbf{V}_1(y_\alpha, \frac{y_3}{\delta})$$

Definition

Suppose that for all y_α, $\|\mathbf{V}(y_\alpha, 0)\| \neq 0$. The *Skin Depth* in a conductor body at conductivity σ and y_α on a Regular Surface Σ is the positive real $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ taking the smallest value s.t.

$$\|\mathbf{V}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \frac{1}{\mathrm{e}}\|\mathbf{V}(y_\alpha, 0)\|$$

Asymptotic Behavior for Skin Depth

Theorem

Let Σ be a Regular Surface with mean curvature $\mathcal{H}$. Suppose that $H_0(y_\alpha) \neq 0$. The Skin Depth $\mathcal{L}(\sigma, y_\alpha)$ has the asymptotic expansion

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\big)$$

MONIQUE DAUGE, ERWAN FAOU, V. PÉRON (2009)

Key of the proof:

$$\|\mathbf{V}(y_\alpha, y_3)\|^2 = \Big[\|H_0\|^2 + 2\delta\,\mathrm{Re}\langle H_0, H_1\rangle + 2y_3\mathcal{H}\,\|H_0\|^2 + \mathcal{O}(\delta, y_3)\Big]\,\mathrm{e}^{-2y_3/\ell(\sigma)}$$

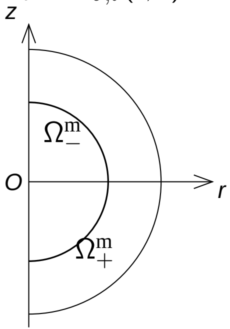
Outline

Numerical Simulations

FEM Discretization

- Magnetic Field
- Framework: Ω_- and Ω Axisymmetric Domains
- **F** Axisymmetric & Orthoradial $\longrightarrow$ $\mathbf{H}_\sigma$ Axisymmetric & Orthoradial

  2D Problem $\longrightarrow$ accuracy
- Variational Form : Unknown $H_\sigma = H_{\sigma,\theta}(r, z) \in V$

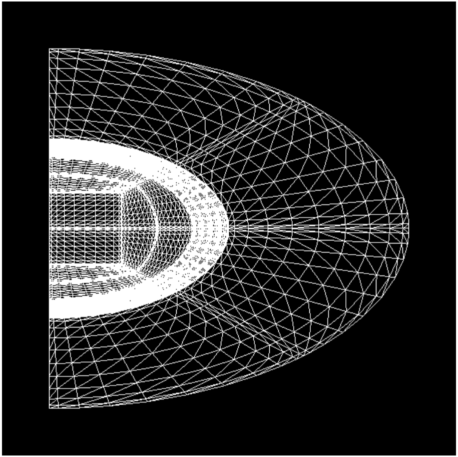


- Finite Element Library Melina

Numerical simulations

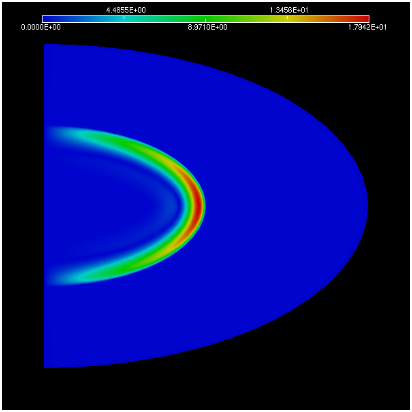
Mesh Refinement

Ω_- & Ω coaxial ellipsoidal domains $\longrightarrow$ Ω_-^{m} & Ω^{m} coaxial half-ellipsis

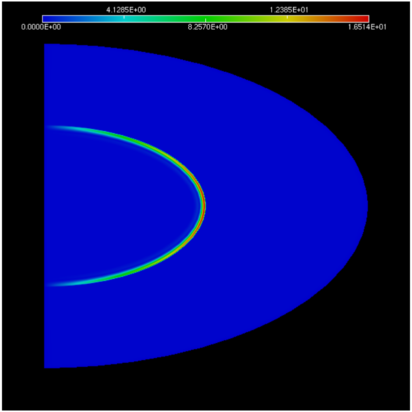


Numerical Simulations

Skin Effect in ellipsoidal geometry



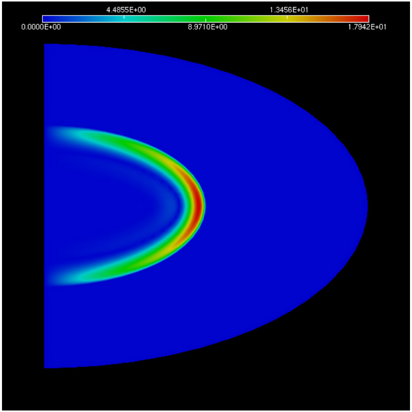
$\left| \operatorname{Im} H_\sigma \right|$ for $\sigma = 5S.m^{-1}$



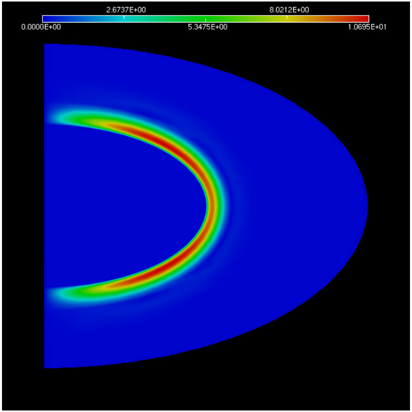
$\left| \operatorname{Im} H_\sigma \right|$ for $\sigma = 80S.m^{-1}$

Asymptotic Behavior for Skin Depth in Ellipsoidal Geometry

$\left|\operatorname{Im} H_\sigma\right|$ for $\sigma = 5S.m^{-1}$



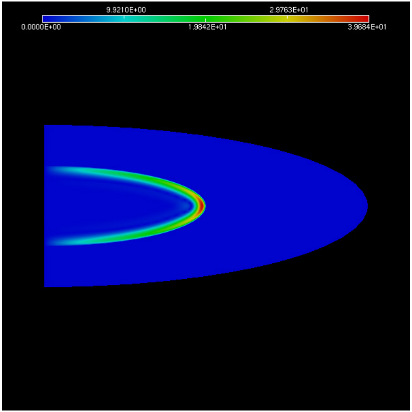
Conductor $= \Omega_-^{\mathrm{m}}$ $(\mathcal{H} > 0)$



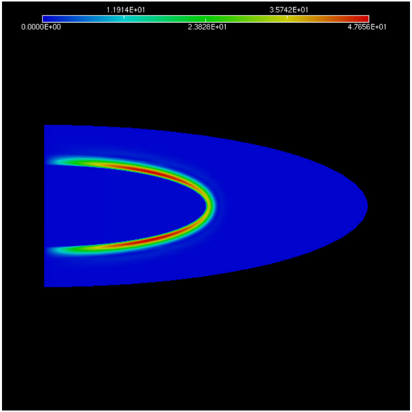
Conductor $= \Omega_+^{\mathrm{m}}$ $(\mathcal{H} < 0)$

Asymptotic Behavior for Skin Depth in Ellipsoidal Geometry

$\left|\operatorname{Im} H_\sigma\right|$ for $\sigma = 5S.m^{-1}$



Conductor $= \Omega^{\mathrm{m}}_{-}$



Conductor $= \Omega^{\mathrm{m}}_{+}$

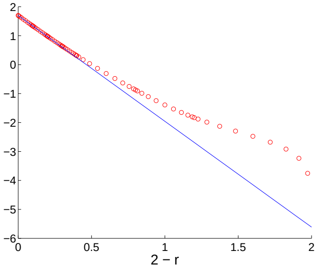
Numerical Post-Treatment

Linear Regression

Extract $\left|H_\sigma(h)\right|$ along edges of the mesh of Ω^{m}_{-} when $z = 0$: $h = 2 - r$.

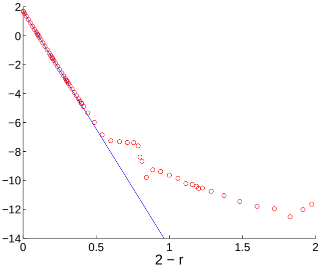
Linear Regression in $skin(\sigma) \longrightarrow n(\sigma)$ points $\longrightarrow$ "numerical" slope $s(\sigma)$

$$\log_{10}\left|H_\sigma(h)\right| = s(\sigma)h + b$$



○ : $\log_{10}\left|H_\sigma(2-r)\right|$ when $\sigma = 5$

$n(\sigma) = 7\,,\, s(\sigma) = -3,65542$



○ : $\log_{10}\left|H_\sigma(2-r)\right|$ when $\sigma = 80$

$n(\sigma) = 3\,,\, s(\sigma) = -16,279162$

Post-Treatment

Accurary of Asymptotics

- Troncated Asymptotic Expansion : $\mathrm{H}^{-}_{\theta,1} := \mathcal{H}^{-}_{\theta,0} + \delta\mathcal{H}^{-}_{\theta,1}$
- $\log_{10}\left|\mathrm{H}^{-}_{\theta,1}(.,2-r)\right| = c + \beta(\sigma,\Sigma)(2-r) + \mathcal{O}(\frac{1}{\sqrt{\sigma}};2-r)$
- "Theoretical" slope $\beta(\sigma,\Sigma) := \frac{1}{\ln 10}\left(\mathcal{H} - \frac{1}{\ell(\sigma)}\right)$
- Relative Error between slopes: $error(\sigma) := \left|\frac{\beta(\sigma,\Sigma)-s(\sigma)}{\beta(\sigma,\Sigma)}\right|$

σ	5	20	80
$skin(\sigma)$ (cm)	10.3	5.15	2.58
$n(\sigma)$	7	5	3
$s(\sigma)$	$-3,65542$	$-7,883903$	$-16,279162$
$\beta(\sigma,\Sigma)$	$-3,673319$	$-7,889506$	$-16,32188$
$error(\sigma)$ (%)	$0,48$	$0,07$	$0,26$